Whispering-gallery-mode resonators as frequency references. I. Fundamental limitations

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Received January 22, 2007; accepted February 11, 2007;
posted March 1, 2007 (Doc. ID 79188); published May 17, 2007

We discuss thermodynamic as well as quantum limitations of the stability of resonance frequencies of solid-state whispering-gallery-mode resonators. We show that the relative frequency stability of a millimeter scale resonator can reach one part per 10^{-12} per 1 s integration time. © 2007 Optical Society of America

OCIS codes: 230.5750, 120.6810, 350.5340.

1. INTRODUCTION

A laser locked to a narrow resonance of a high-finesse resonator is a common source of stable narrow-linewidth optical signals.[1–6] The frequency stability of the resonator mode, generally limited by mechanical[7,8] as well as temperature fluctuations,[9] determines the laser stability. Whispering-gallery-mode (WGM) resonators, featuring small size, high transparency windows, and narrow resonances, are natural candidates for use in laser stabilization.[10] These resonators also can be helpful in microwave photonics, for instance for frequency stabilization of optoelectronic[11] and coupled optoelectronic oscillators.[12] The possibility of direct generation of high-stability microwave signals using various nonlinear processes enhanced due to the high-quality factors and small-mode volumes of the resonators[13–17] should also be considered. In this paper, we evaluate the thermodynamic limitations of the frequency stability of WGM resonators. In a future paper, we will discuss the possibility of stabilization of WGM frequency beyond the thermodynamic limit found in this paper. Finally, in a third paper of the series, we will study the performance limitations of various frequency references stabilized with WGM resonators.

WGM optical resonators are dielectric structures that support extremely high-quality (Q-) factor modes propagating close to the surface of the resonator. The ultimate Q factors of the modes are determined by the intrinsic material loss and scattering. The attenuation is minimized for resonators produced with crystalline materials.[18] Crystalline WGM resonators have ultra-high-Q factors in a wide wavelength range, and some crystalline optical materials such as calcium fluoride have small absorption and scattering in a broad wavelength region. According to recently reported measurements,[19] the light attenuation coefficient of CaF_2 is $6\times10^{-5}\,\mathrm{cm}^{-1}$ at 157 nm and even lower in visible and infrared wavelength ranges, which gives us grounds to believe that the Q factor of a WGM resonator made out of calcium fluoride will exceed 10^9 in the frequency range spanning from 150 to 8000 nm.

Crystalline WGM resonators have many other attractive features. They are generally immune to atmospheric humidity, unlike, e.g., resonators made with fused silica that have a degraded Q factor in humid air. The resonator-based devices are extremely compact. There are several techniques for efficient in and out light coupling with crystalline resonators. A great advantage of a WGM resonator is that it is possible to engineer its modal spectrum via a modification of the shape of their surface.

The major advantages of WGM resonators compared with Fabry–Perot (FP) resonators commonly used as frequency references are: (i) the small size of WGM resonators ranging from a few hundred micrometers to a few millimeters; (ii) a large wavelength range where WGM resonators have high Q factors; and (iii) the low sensitivity of WGM resonators to mechanical noise because of the unique orthogonality relations between optical and acoustical WGM modes (triply resonant scattering "photon $\leftrightarrow$ photon and phonon" is generally suppressed), and because of the high-Q factors and the high frequency of the acoustical modes. All these features make utilization of the WGM resonators in place of FP resonators attractive for a broad range of applications.

On the other hand, WGM resonators have several disadvantages compared with FP resonators. (i) Reference FP resonators contain a specific mirror spacer material that has a low thermal expansion coefficient. WGM resonators cannot contain such a material. The polished rim surface of the WGM resonator plays the role of the resonator mirrors, and the host material is the only resonator mirror spacer material possible. Such a material generally has a large thermal coefficient of expansion. This problem can be mediated by efficient thermal isolation of the compact WGM resonators, which is simpler than the isolation of FP resonators because of the small size and structure rigidity of the WGM resonators. (ii) Reference FP resonators are empty, i.e., devoid of any material, and are comparably large. Those properties reduce the fundamental thermodynamic fluctuations, which are important even if the resonator is kept at a constant temperature. WGM resonators are small and material filled and hence

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suffer more strongly from the thermodynamic fluctuations.[20] (iii) The optical nonlinearity of WGM resonators is much higher than the nonlinearity of FP resonators because of the same reasons as indicated in the previous item. The circulating optical power level should be limited in a WGM resonator to mediate this problem. The power fluctuations of a laser interrogating the resonator should also be small.

The goal of the present contribution is elucidating the properties of WGM frequency fluctuations, resulting from the basic fundamental thermodynamic as well as quantum optic principles. We evaluate the frequency spectra of thermorefractive and thermoelastic fluctuations, as well as the steady-state WGM frequency uncertainty resulting from those fluctuations, both numerically and analytically. We show that both types of thermodynamic fluctuations contribute equally to the steady-state frequency uncertainty. On the other hand, the spectral power density of the frequency noise is given primarily by the thermorefractive fluctuations. We also study the photothermal and ponderomotive fluctuations originating from the measurement procedure and find their frequency spectra.

We show that the stability of WGM frequency is primarily determined by the thermorefractive fluctuations. For instance, the fluctuations result in a one-second-averaged Allan deviation of 10^{-12} in the WGM frequency of a calcium fluoride resonator with a 0.3 cm radius and a 0.01 cm thickness. The deviations originating from other fluctuations are at least 2 orders of magnitude smaller.

This paper is organized as follows. The fundamental thermodynamic fluctuations of a WGM resonator are studied in Section 2. The fluctuations originating from the measurement procedure are studied in Section 3. Section 4 concludes the paper. All extensive calculations can be found in Appendices A–F.

2. FUNDAMENTAL THERMODYNAMIC FLUCTUATION OF A WHISPERING-GALLERY MODE

In this section, we consider two types of resonators. We initially assume that a WGM resonator can be considered as a small part of a much larger object, so the thermal spectrum is continuous. This assumption is valid, for instance, for a low contrast resonator[21] or for a resonator being in good contact with the heat bath. The validity of the assumption is even more general. It was shown in Ref. 20 that it is possible to neglect the discreteness of the spectrum of the thermal waves in a sufficiently large spherical resonator (radius exceeding several tens of micrometers) because the mode volume is small compared with the volume of the resonator. To confirm that conclusion, we also consider a thin resonator, so the discrete thermal spectrum should be studied. We show that the resonators with both discrete and continuous thermal spectra possess equivalent frequency fluctuations.

As a general rule, the relative uncertainty of the eigenfrequency of WGMs of a thin cylindrical (toroidal) resonator can be found from the expression

$$\frac{\delta\omega}{\omega} = \frac{\Delta R}{R} + \frac{\Delta n}{n}, \tag{1}$$

where ω is the mean value of the frequency of a selected mode, $\delta\omega$ is the fluctuation of the frequency, R and ΔR are the values of the radius of the resonator and its fluctuation, and n and Δn are the values of the index of refraction of the material of the resonator and its fluctuation. Equation (1) is valid for a WGM resonator of any relevant shape if the radius of the resonator determines the largest dimension of the geometrical localization of the mode, i.e., where $\omega_{WGM} \simeq c\nu/(Rn)$, where $\nu \gg 1$ is the azimuthal mode number.

Generally, the terms $\Delta R/R$ and $\Delta n/n$ in Eq. (1) are responsible for thermoelastic and thermorefractive noise, respectively; however a correlation between them is possible. Both terms are of the same order, and they both should be taken into account in crystalline WGM resonators.

A. Thermorefractive Fluctuations: Steady State

Thermodynamic fluctuation of temperature in the WGM volume results in fluctuations of the index of refraction in the WGM channel,

$$\left.\frac{\Delta n}{n}\right|_{(\Delta T)_m} = \alpha_n (\Delta T)_m, \tag{2}$$

where $\alpha_n = (1/n)(\partial n/\partial T)$ is the thermorefractive coefficient of the resonator host material, and $(\Delta T)_m$ is the fluctuation of the temperature averaged over the mode volume relative to the averaged temperature of the whole system. The mean-square value of the thermal fluctuation is[22]

$$\langle (\Delta T)_m^2 \rangle = \frac{k_B T^2}{C_p V_m \rho}, \tag{3}$$

where k_B is Boltzmann's constant, T is the absolute temperature, ρ is the density of the resonator host material, V_m is the mode volume (assumed to be much less than the volume of the resonator), and C_p is the specific heat capacity at constant pressure of the resonator host material. In what follows, we assume that $C_p = C_V = C$ for a crystalline material. The mode volume of a WGM belonging to the basic mode sequence of a spherical resonator is[23]

$$V_m = 3.4\pi^{3/2}\left(\frac{\lambda}{2\pi n}\right)^{7/6} R^{11/6}. \tag{4}$$

The value of the mode volume of a toroidal and/or cylindrical WGM resonator varies, depending on the resonator thickness; it can exceed the value for the spherical resonator.

B. Thermorefractive Fluctuations: Spectrum

The frequency spectrum of thermorefractive fluctuations can be found following the path described in Ref. 20. The complex temperature distribution u in the resonator is given by the heat transport equation containing a distributed external fluctuational thermal source,[24]

$$\frac{\partial u}{\partial t} - D\Delta u = F(\mathbf{r},t), \tag{5}$$

where $D=\kappa/(\rho C)$, κ is the thermal conductivity coefficient, and C is the specific heat capacity. The thermal source is normalized such that the quadratic deviation of the mode temperature coincides with expression (3) [see, e.g., Eq. (B20) as well as a discussion in Appendices A and B].

We are interested in the temperature fluctuations averaged over the mode volume,

$$\bar{u}(t) = \int u(\mathbf{r},t)|\Psi(\mathbf{r})|^2 \mathrm{d}\mathbf{r}, \tag{6}$$

where $|\Psi(\mathbf{r})|^2$ is the normalized spatial power distribution for the light localized in a WGM, $\int|\Psi(\mathbf{r})|^2\mathrm{d}\mathbf{r}=1$.

The spectral power density of the random process $\bar{u}(t)$ and the quadratic deviation of the average mode temperature $\langle u(0)^2\rangle$ are given by

$$S_{\bar{u}}(\Omega) = \int_{-\infty}^{\infty} \langle \bar{u}^*(t)\bar{u}(t+\tau)\rangle e^{-i\Omega\tau}\mathrm{d}\tau, \tag{7}$$

$$\langle(\Delta T)_m^2\rangle = \langle \bar{u}(0)^2\rangle = \int_{-\infty}^{\infty} S_{\bar{u}}(\Omega)\frac{\mathrm{d}\Omega}{2\pi}. \tag{8}$$

Let us consider a resonator formed on the surface of an infinitely long cylinder of radius R by a cylindrical protrusion of radius $R+\Delta R$ ($R \gg \Delta R$) and thickness L. We solve Eq. (5) and numerically evaluate the spectral density of frequency fluctuations for a calcium fluoride resonator with $R=0.3$ cm and $L=0.01$ cm [see Appendix A and Fig. 1].

We also find the fluctuations for a thin cylindrical resonator of thickness L and radius R, $R \gg L$. A simple analytical approximation is possible in this case. We present the correlation of the fluctuation forces as

$$\langle F(\mathbf{r},t)F(\mathbf{r}',t')\rangle \simeq 16\pi \frac{k_B T^2 D L}{\rho C V_m}\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'); \tag{9}$$

solving Eq. (5), we derive an approximate expression for the spectral density of the thermorefractive noise (see Appendix B):

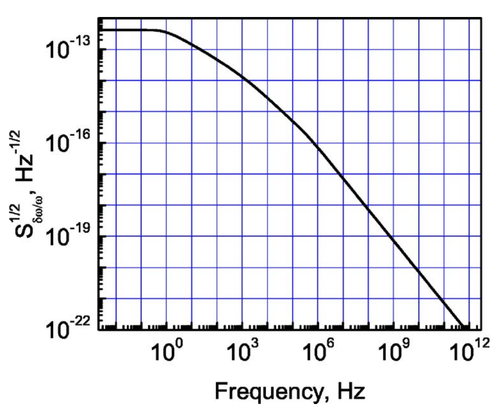

Fig. 1. (Color online) Spectral power density of the frequency noise $S_{\delta\omega/\omega}(\Omega)=\alpha_n^2 S_{\bar{u}}(\Omega)$ resulting from the thermorefractive fluctuations in a low-contrast WGM resonator.

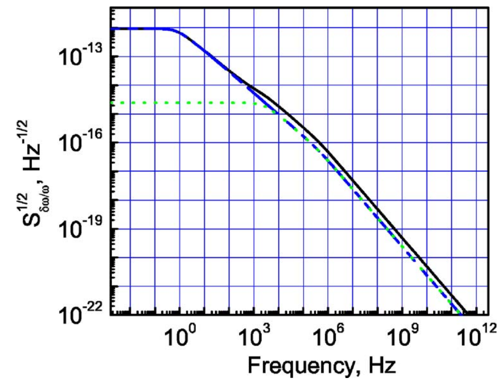

Fig. 2. (Color online) Spectral density of the frequency noise $S_{\delta\omega/\omega}(\Omega)=\alpha_n^2 S_{\bar{u}}(\Omega)$. The solid (black) curve is derived from the exact solution given by Eq. (B19), found by the summation of contributions of modes with $m=0$ (dashed blue curve) and $m=2$ (dotted green curve).

$$S_{\bar{u}}(\Omega) = \frac{k_B T^2}{\rho C V_m}\frac{R^2}{12D} \times \left[1 + \left(\frac{R^2}{D}\frac{|\Omega|}{9\sqrt{3}}\right)^{3/2} + \frac{1}{6}\left(\frac{R^2}{D}\frac{\Omega}{8\nu^{1/3}}\right)^2\right]^{-1}. \tag{10}$$

A comparison of this approximate solution with an exact solution for a particular example is given in Appendix B. The spectral density of the frequency noise, given by $S_{\delta\omega/\omega}(\Omega)=\alpha_n^2 S_{\bar{u}}(\Omega)$, is approximately the same for the case of a thin resonator and for a resonator being a part of an infinite cylinder [compare Figs. 1 and 2].

C. Thermoelastic Fluctuations: Steady State

Note that the variations of the mode volume of a WGM due to, e.g., fundamental thermoelastic fluctuations, do not change the WGM frequency. The frequency is given by the boundary conditions of the resonator and by the value of its radius. The fluctuation of the radius, in turn, is determined by the fluctuation of the volume as well as the temperature of the entire resonator. We show that the fluctuations of the temperature averaged over the resonator volume are small compared with the temperature fluctuations averaged over the mode volume and, hence, can be neglected. The thermoelastic fluctuations should be taken into account.

To use the thermodynamic approach, we assume that the resonator is a small part of a much larger sample. This is possible if the resonator is a thin disk cut on a solid-state rod. Then the volume of the resonator V_r fluctuates as[22]

$$\frac{\langle(\Delta V_r)^2\rangle}{V_r^2} = k_B T\frac{\beta_T}{V_r}, \tag{11}$$

and $\beta_T=-[(1/V)(\partial V/\partial p)]_T$ is the isothermal compressibility of the resonator host material. Fluctuation of the radius of a spherical resonator due to fluctuations of its volume is

$$\left.\frac{\Delta R}{R}\right|_{\Delta V} = \frac{\Delta V_r}{3V_r}. \tag{12}$$

Fluctuation of the average resonator temperature,

$$\langle(\Delta T)_r^2\rangle = \frac{k_B T^2}{C_p V_r \rho}, \tag{13}$$

influences the radius as well:

$$\left.\frac{\Delta R}{R}\right|_{\Delta T} = \alpha_l (\Delta T)_r, \tag{14}$$

where $\alpha_l = (1/l)(\partial l/\partial T)$ is the linear thermal expansion coefficient.

Using the fact that fluctuations $(\Delta V)_r$ and $(\Delta T)_r$ are statistically independent, and that the correlation of fluctuations of the temperature of the whole resonator and the average temperature in the WGM localization, $(\Delta T)_r$ and $(\Delta T)_m$, respectively, is small (mode volume is much smaller than the volume of the resonator), we arrive at

$$\frac{\langle(\Delta R)^2\rangle}{R^2} = k_B T \frac{\beta_T}{9V_r} + \alpha_l^2 \frac{k_B T^2}{C_p V_r \rho}. \tag{15}$$

Let us compare the influence of the fundamental thermorefractive and thermoelastic fluctuations on the WGM frequency. Because $|\alpha_l|$ and $|\alpha_n|$ are usually comparable for crystalline materials and $V_r \gg V_m$, the second term on the right-hand side of Eq. (15) is much smaller as compared with the thermorefractive term given by Eq. (2). The first term on the right-hand side of Eq. (15) is significant. This should also be compared with the squared Eq. (2). For instance, for a cylindrical fluorite resonator with $R = 0.3$ cm and $L = 0.01$ cm at room temperature, the thermoelastic term is two times smaller than the thermorefractive term. Therefore, both the thermoelastic and thermorefractive terms should be taken into account to find the frequency uncertainty of a WGM.

It is important to note that the thermoelastic noise depends on the shape of the resonator. For instance, the thermoelastic fluctuations of a thin cylindrical resonator mounted on a thin stem should be considered in a different way compared with the above derivation. The above estimate is valid only if there is at least one continuous (quasi-continuous) dimension in the system, for instance, if the resonator is formed by a small protrusion on a long cylindric rod. We also show in Appendix D that thermal radial oscillations of a liquidlike sphere result in a thermoelastic fluctuational term similar to the one in Eq. (15).

Finally, we should mention that usually only temperature fluctuations transformed into volume fluctuations through thermal expansion [Eq. (14)] are called thermoelastic. The thermodynamic fluctuations of volume [Eq. (12)] are called Brownian.

D. Thermoelastic Fluctuations: Spectrum

We have shown in the previous section that thermoelastic fluctuations should be taken into account to determine the frequency stability of a small crystalline resonator. Let us find the spectral density of the frequency noise determined by the fluctuations. For the sake of simplicity, we consider only one effective 1D elastic mode of the resonator of the lowest order. The contributions from other modes of higher order are comparably small. We assume that in the vicinity of the WGM localization, the radius of the resonator changes in accordance with

$$\frac{\partial(\Delta R)}{\partial t} + (-i\Omega_0 + \Gamma_0)\Delta R = F_R(t), \tag{16}$$

where the oscillation frequency is taken to be equal to the eigenfrequency of the lowest-order radial mode of a spherical resonator of radius R,

$$\Omega_0 = \frac{\pi v_s}{R}, \tag{17}$$

where v_s is the speed of sound. We do not consider the other types of mechanical oscillations of the resonator.

The decay rate of the acoustic mode can be very small. The minimal value of the rate is thermodynamically limited[25]:

$$\Gamma_0 \geq \frac{\Omega_0^2 \kappa T \alpha_l^2 \rho}{9C^2}. \tag{18}$$

This is a very small value. The realistic value of the quality factor $(\Omega_0/2\Gamma_0)$ of the acoustic mode is expected to exceed 5×10^4 (Ref. 9).

We select the fluctuational force $F_R(t)$ such that it obeys

$$\langle F_R^*(t)F_R(t')\rangle = \Gamma_0 k_B T \frac{\beta_T R^2}{9V_r}\delta(t-t'), \tag{19}$$

and obtain an expression for the spectral density of the radius fluctuation:

$$S_{\Delta R/R} = k_B T \frac{\beta_T}{9V_r}\frac{\Gamma_0}{(\Omega-\Omega_0)^2+\Gamma_0^2}. \tag{20}$$

The density is peaked at Ω_0 and is significantly suppressed at higher frequencies. For example, for a cylindrical resonator with $R=0.3$ cm, $L=0.01$ cm, and $v_s=5\times10^5$ cm/s, we have $\Omega_0=5\times10^6$ rad/s. Assuming that $\Gamma_0=100$ rad/s, we obtain Fig. 3. It is easy to see that the low-frequency branch of the spectral power density of the frequency noise given by the thermoelastic fluctuations ($S_{\delta\omega/\omega}=S_{\Delta R/R}$) is much smaller than the one due to thermorefractive fluctuations.

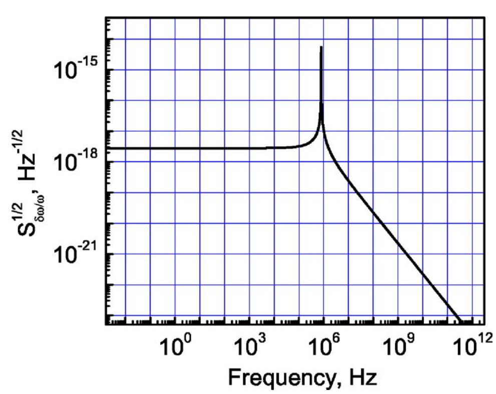

Fig. 3. (Color online) Spectral density of the frequency noise $S_{\delta\omega/\omega}(\Omega)=S_{\Delta R/R}(\Omega)$ due to thermoelastic noise.

Table 1. Parameters of the Calcium Fluoride, Sapphire, and Fused Silica

Parameter	CaF_2	Al_2O_3	SiO_2
ρ, g/cm^3	3.18	4.0	2.2
β_T, 10^{-12} cm^2/dyn	1.2	0.4	2.7
C_p, 10^6 erg/(g K)	8.54	7.61	6.7
n at 1.55 μm	1.42	1.75(o)/1.74(e)	1.46
α_l, 10^{-5} K^{-1}	1.89	5.4(o)/6.2(e)	5.5×10^{-2}
α_n,10^{-5} K^{-1}	−0.75	1.0	1.0
κ, 10^5 erg/(cm s K)	9.7	24	1.4

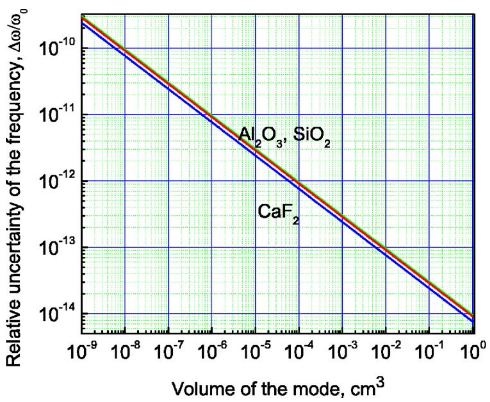

Fig. 4. (Color online) Thermodynamic uncertainty of the absolute frequency of a WGM versus mode volume calculated for a spherical resonator.

E. Fundamental Frequency Uncertainty

Let us estimate how the relative frequency uncertainty of a WGM depends on the averaged mode volume. Substituting the data presented in Table 1 as well as $k_B=1.38\times 10^{-16}$ erg/K, $T=300$ K, into Eqs. (1), (2), and (15) we obtain Fig. 4. It is easy to see that the frequency uncertainty almost does not depend on the material the WGM resonator is made from. The reasonable value of the relative frequency uncertainty for existing crystalline WGM resonators[18] is of the order of 10^{-12}.

3. FLUCTUATIONS ORIGINATING FROM THE MEASUREMENT PROCEDURE

In this section, we consider fluctuations of the WGM frequency arising from the measurement procedure. The resonators are interrogated with laser radiation. Shot noise of the radiation causes two additional types of fluctuations: photothermal,[24,26] optoelastic,[27] and self-phase modulational.

A. Photothermal Fluctuations

These fluctuations appear as the result of the transfer of the shot noise of light absorbed in the resonator to the temperature fluctuations of the resonator host material and to the subsequent fluctuations of the index of refraction of the resonator.

The temperature distribution in the resonator is described by the equation,

$$\frac{\partial u}{\partial t} - D\Delta u = F_P(\mathbf{r},t), \tag{21}$$

where $F_P(\mathbf{r},t)$ is the fluctuational force describing noise due to the absorption of the photons in the resonator,

$$\langle F_P(\mathbf{r},t)F_P(\mathbf{r}',t')\rangle = \frac{\hbar\,\omega_{\nu,q,m}}{\rho^2 C^2}\langle P_{abs}\rangle |\Psi(\mathbf{r})|^2 \delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{22}$$

$\omega_{\nu,q,m}$ is the angular frequency of the corresponding WGM, $\langle P_{abs}\rangle$ is the expectation value for the absorbed power, and $|\Psi(\mathbf{r})|^2$ is the power distribution in a WGM ($\int |\Psi(\mathbf{r})|^2 d\mathbf{r}=1$).

We find (see Appendix E) that the spectral density of the temperature fluctuations for a cylindric resonator can be approximated by

$$S_{\bar{u}}(\Omega) \approx \frac{\hbar\,\omega_{\nu,q,m}}{\rho^2 C^2}\frac{\langle P_{abs}\rangle}{V_r^2}\frac{\pi^2 R^{3/2}/64L^{3/2}}{\Omega^2+\pi^6 D^2/16L^3 R}. \tag{23}$$

A comparison of the exact numerical simulation and the approximation of the spectral density is presented in Fig. 5. It is easy to see that the photothermal fluctuations are small for reasonable values of the absorbed optical power.

B. Ponderomotive Fluctuations

Ponderomotive fluctuations occur as a result of the fluctuations of the radiation pressure induced by light propagating inside the resonator.[25,28] The integral value of the pressure induced force is $F=2\pi Pn/c$, where P is the power of the light inside the resonator. The force changes the resonator radius as well as the index of refraction. The index of refraction is involved due to the mechanical strain of the resonator host material:

$$\frac{\delta\omega}{\omega} = \left[1+\frac{1}{2}K_\varepsilon\right]\frac{\Delta R}{R}, \tag{24}$$

where factor $K_\varepsilon=-E\varepsilon^{-1}\partial\varepsilon/\partial p$ ranges from 1 to 10,[25] E is Young's modulus of the material, and $\varepsilon=n^2$ is the electric susceptibility of the material.

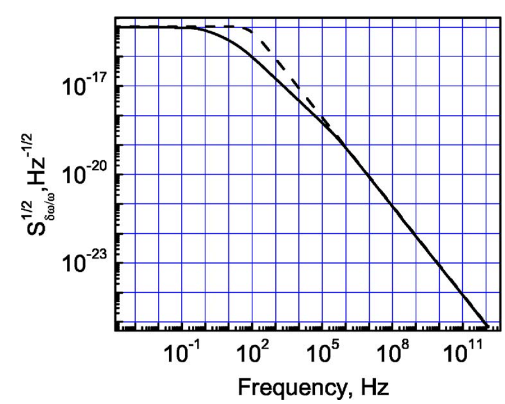

Fig. 5. (Color online) Spectral density of the photothermal fluctuations of a WGM frequency ($S^{1/2}_{\delta\omega/\omega}=\alpha_n S^{1/2}_{\bar{u}}$) calculated for a fluorite resonator with $R=0.3$ cm and $L=0.01$ cm interrogated with coherent 1.55 μm light of 1 mW power assuming that the light is absorbed in the resonator. The solid (dashed) curve stands for the simulation (analytical calculations).

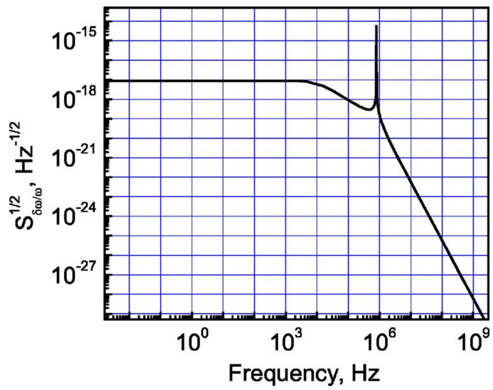

Fig. 6. (Color online) Spectral density of the ponderomotive backaction fluctuations of frequency $[S^{1/2}_{\delta\omega/\omega}=(1+K_\epsilon/2)S^{1/2}_{\Delta R/R}]$ calculated for a fluorite resonator with $R=0.3$ cm and $L=0.01$ cm interrogated with 1 mW coherent light.

For the sake of simplicity, we take into account only one mechanical mode having the lowest mechanical frequency (17) and assume that the probe light is resonant with the corresponding WGM. The last condition is required to avoid mechanical instability or additional rigidity added to the mechanical system by light. The mechanical oscillations of the resonator surface are described by the equation

$$\frac{\partial^2(\Delta R)}{\partial t^2}+\Gamma_0\frac{\partial(\Delta R)}{\partial t}+\Omega_0^2(\Delta R)=\frac{2\pi n\,\delta P(t)}{m^* c}, \qquad (25)$$

where $m^*\approx\rho V_r$ is the effective mass of the oscillator, and $\delta P(t)$ is the variation of the optical power in the corresponding WGM. Calculating $\delta P(t)$ using the general Langevin formalism (see Appendix F) we derive an expression for the spectral density of the fluctuations:

$$S_{\Delta R/R}=\left(\frac{2\pi n}{m^* cR}\right)^2\frac{\hbar\,\omega_{\nu,q,m}\langle P\rangle}{\tau_0}\times\frac{2\gamma_R}{\gamma_R^2+\Omega^2}\frac{1}{(\Omega_0^2-\Omega^2)^2+\Gamma_0^2\Omega^2}, \qquad (26)$$

where γ_R is the spectral width of the mode. Because generally $\Omega_0\gg\gamma_R, \Gamma_0$ we find an expression for the square deviation of the radius of the resonator resulting from the fluctuations of the radiation pressure:

$$\frac{\langle\Delta R^2(t)\rangle^{1/2}}{R}\simeq\frac{2\pi n}{m^* cR}\left[\frac{\hbar\,\omega_{\nu,q,m}\langle P\rangle}{\tau_0\Omega_0^4}\left(1+\frac{\gamma_R}{\Gamma_0}\right)\right]^{1/2}. \qquad (27)$$

Let us estimate the value for a fluorite resonator with $R=0.3$ cm and $L=0.01$ cm interrogated with coherent 1.55 μm light of 1 mW input power. We also assume that $\gamma_R=2\pi\times10^4$ rad/s and $\Gamma_0=100$ rad/s. We find the averaged power inside the resonator $\langle P\rangle=1\text{ mW}\times[2/\tau_0\gamma_R]\simeq340$ W, square deviation of the radius $(\langle\Delta R^2(t)\rangle/R^2)^{1/2}\simeq8\times10^{-16}$, and low-frequency spectral density $(S_{\Delta R/R}(0))^{1/2}=4\times10^{-18}\text{ Hz}^{-1/2}$. The corresponding spectral density of frequency fluctuations for $K_\varepsilon=4$ is shown in Fig. 6.

Our calculations show that the radiation pressure fluctuations are comparably weak in a sufficiently large WGM resonator and can be neglected in the majority of cases when the resonator is interrogated with low-power light.

C. Self-Phase-Modulational Fluctuations

Fluctuations resulting from Kerr nonlinearity of the resonator host material result in a variation of the refractive index of the material,

$$\frac{\delta n}{n}=\frac{n_2\delta P(t)}{\mathcal{A}n}, \qquad (28)$$

where n_2 is the Kerr nonlinearity of the resonator and $\mathcal{A}\equiv V_m/2\pi R$ is the cross-section area of the WGM. The change of the refractive index results in the shift of the WGM frequencies.

Using the results of the previous section, we obtain the following expression for the spectral density,

$$S_{\delta\omega/\omega}=\frac{n_2^2\hbar\,\omega_{\nu,q,m}\langle P\rangle}{\mathcal{A}^2n^2\tau_0}\frac{2\gamma_R}{\gamma_R^2+\Omega^2}, \qquad (29)$$

and frequency uncertainty,

$$\frac{\delta\omega}{\omega}=\frac{n_2\sqrt{\hbar\,\omega_{\nu,q,m}\langle P\rangle}}{\mathcal{A}n\sqrt{\tau_0}}. \qquad (30)$$

We estimate the values for a fluorite resonator with $R=0.3$ cm and $L=0.01$ cm with $\gamma_R=2\pi\times10^4$ rad/s interrogated with coherent 1.55 μm light of 1 mW input power. The averaged power inside the resonator is $\langle P\rangle\simeq340$ W, the cross-section area is $\mathcal{A}=3\times10^{-6}\text{ cm}^2$ (see Appendix C), $\tau_0=2\pi Rn/c$, and $n=1.45$. The nonlinearity of the material is $n_2=3.2\times10^{-16}\text{ cm}^2/\text{W}$. Finally, we find $\delta\omega/\omega\simeq5\times10^{-14}$. The spectral density of the frequency fluctuations originating from the effect of self-phase modulation is shown in Fig. 7.

In practice, the power should be much lower to allow operation below the stimulated Raman scattering as well as the four-wave-mixing threshold. Hence, the frequency uncertainty originating from the optical nonlinearity of the resonator host material is small enough.

4. SUMMARY AND CONCLUSION

In this paper, we have studied the fundamental noise sources that affect the frequency stability of WGM resonators and have found spectral power densities of fluctua-

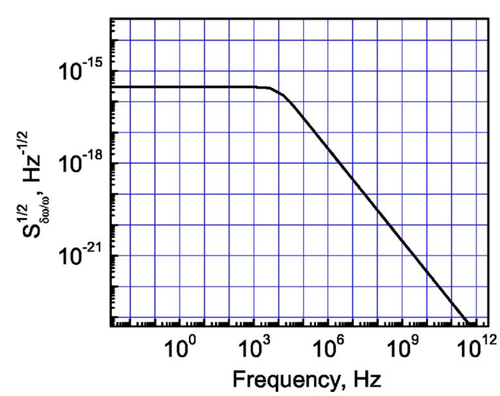

Fig. 7. (Color online) Spectral density of the frequency noise $S^{1/2}_{\delta\omega/\omega}(\Omega)$ resulting from the self-phase-modulation effect in the CaF_2 resonator. The noise is calculated for a fluorite resonator with $R=0.3$ cm and $L=0.01$ cm interrogated with 1 mW coherent light.

tions of frequency for such resonators. The fluctuations are caused by thermorefractivity, thermoelasticity, and the ponderomotive effect of light. We have shown that the thermorefractive fluctuations are the major sources contributing to the frequency fluctuations in a WGM resonator interrogated with low-power light.

Nevertheless, our findings indicate that the fluctuations are small enough to make the resonators attractive as secondary frequency references. It is convenient to use the Allan variance of the WGM frequency to quantify the frequency stability. We can find the variance by integrating the evaluated spectral density of the fluctuations[29]:

$$\sigma^2(\tau) = \frac{2}{\pi}\int_0^\infty S_{\delta\omega/\omega}(\Omega)\frac{\sin^4(\Omega\tau/2)}{(\Omega\tau/2)^2}\mathrm{d}\Omega, \tag{31}$$

where we took into account that $S_{\delta\omega/\omega}(\Omega)$ is a double-sided spectral density. We find that the thermorefractive fluctuations [see Eq. (10)] result in $\sigma(\tau=1\,\mathrm{s})\simeq 10^{-12}$. The value can be made even smaller by further averaging and increasing the integration time. In the second paper of this series, we will discuss the possibility of suppression of the fundamental thermodynamic fluctuations discussed in this paper.

APPENDIX A: SOLUTION OF THE STOCHASTIC HEAT TRANSFER EQUATION FOR A LOW-CONTRAST RESONATOR

The complex eigenfunctions of the heat transfer equation with zero Neumann boundary condition,

$$\frac{\partial u}{\partial t} - D\Delta u = 0, \tag{A1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \tag{A2}$$

for an infinite cylinder with radius R can be presented in form,

$$u(t) = \sum_{p,l}\int_{-\infty}^{\infty}\tilde{u}_{p,l,k_m}(t)J_l(k_{l,p}r)e^{\pm il\phi}e^{ik_m z}\frac{\mathrm{d}k_m}{2\pi}, \tag{A3}$$

where $J_l(k_{l,p}r)$ is the Bessel function of the first kind, and $k_{l,p}$ are roots of the equation $\partial J_l(k_{l,p}r)/\partial r|_R=0$. Equation (A1) describes the temperature modes of a WGM resonator near thermal equilibrium. Only modes with $l=0$ should be taken into account if we are interested in a study of thermal properties of optical WGMs with field distribution given by

$$\Psi(\mathbf{r}) = \sin\left(\frac{\pi z}{L}\right)\sqrt{\frac{2}{\pi}}\frac{J_\nu(k_{\nu,q}r)}{J_{\nu+1}(k_{\nu,q}R)}\frac{e^{\pm i\nu\phi}}{R\sqrt{L}}, \tag{A4}$$

where $k_{\nu,q}$ are roots of the equation $J_\nu(k_{\nu,q}R)=0$ and $\int|\Psi(\mathbf{r})|^2\mathrm{d}\mathbf{r}=1$. All other temperature, modes have zero contribution to the average temperature of the WGM channel [see Eq. (6)].

Hence, we are looking for a solution of Eq. (5) having the form,

$$u(t) = \sum_p\int_{-\infty}^{\infty}\Phi_{p,k_m}(\mathbf{r})\tilde{u}_{p,k_m}(t)\frac{\mathrm{d}k_m}{2\pi}, \tag{A5}$$

$$\int\Phi_{p,k_m}\Phi^*_{p',k'_m}\mathrm{d}\mathbf{r} = 2\pi\delta(k_m-k'_m)\delta_{p,p'}. \tag{A6}$$

The space-dependent parts of the function are

$$\Phi_{p,k_m}(\mathbf{r}) = \frac{J_0(k_p r)}{J_0(k_p R)}\frac{\exp(ik_m z)}{R\sqrt{\pi}}, \tag{A7}$$

where $k_p=\beta_p/R$, and β_p is the pth root of equation $J_1(\beta_p)=0$. We used the fact that $J_0'(r)=-J_1(r)$. It is also important to note that $\beta_p\simeq\pi p$ if $p\gg 1$.

Substituting Eq. (A5) into Eq. (5), multiplying the equation by $\Phi^*_{p',k'_m}(\mathbf{r})$, and averaging over the volume, we obtain

$$\frac{\partial\tilde{u}_{p,k_m}(t)}{\partial t} + Dk^2_{p,m}\tilde{u}_{p,k_m}(t) = \mathcal{F}_{p,k_m}(t), \tag{A8}$$

where $k^2_{p,m}=k^2_p+k^2_m$ and

$$\langle\mathcal{F}_{p,k_m}(t)\mathcal{F}^*_{p',k'_m}(t')\rangle = 4\pi^2 DR\frac{k_B T^2}{C_p V_m\rho}\delta(t-t')\delta(k_m-k'_m)\delta_{p,p'}. \tag{A9}$$

To find $\bar{u}(t)$ [see Eq. (6)], we note that

$$\int\Phi_{p,k_m}(\mathbf{r})|\Psi(\mathbf{r})|^2\mathrm{d}\mathbf{r} \simeq \frac{8\pi^2\sin(k_m L/2)}{k_m L(k_m^2L^2-4\pi^2)}\frac{\exp(ik_m L/2)}{\sqrt{\pi}R}, \tag{A10}$$

where we have assumed that the radial temperature distribution function can be taken out of the integral if the temperature does not change significantly at a range comparable with the spatial distribution of the WGM field [in other words, we assume that $J_0(k_p r)\simeq J_0(k_p R)$ in the vicinity of the WGM localization].

The spectral power density is

$$S_{\bar{u}}(\Omega) = \frac{1}{\pi R^2}\sum_{p\le\nu^{2/3}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_m}{2\pi}\frac{\mathrm{d}k'_m}{2\pi} \times\left[\frac{8\pi^2\sin(k_m L/2)}{k_m L(k_m^2L^2-4\pi^2)}\right]^2 \times\int_{-\infty}^{\infty}\langle\tilde{u}^*_{p,k_m}(t)\tilde{u}_{p,k'_m}(t+\tau)\rangle e^{-i\Omega\tau}\mathrm{d}\tau. \tag{A11}$$

To find the spectral density, we solve Eq. (A8) using the Fourier transform. We decompose the time-dependent variables as follows:

$$\tilde{u}_{p,k_m}(t) = \int_{-\infty}^{\infty} \tilde{u}_{p,k_m}(\Omega) e^{i\Omega t} \frac{d\Omega}{2\pi}, \tag{A12}$$

$$\mathcal{F}_{p,k_m}(t) = \int_{-\infty}^{\infty} \mathcal{F}_{p,k_m}(\Omega) e^{i\Omega t} \frac{d\Omega}{2\pi}, \tag{A13}$$

where

$$\langle \mathcal{F}^*_{p,k_m}(\Omega) \mathcal{F}_{p',k'_m}(\Omega') \rangle = 8\pi^3 DR \frac{k_B T^2}{C_p V_m \rho} \delta(\Omega - \Omega') \delta(k_m - k'_m) \delta_{p,p'}. \tag{A14}$$

The solution of Eq. (A8) is

$$\tilde{u}_{p,k_m}(t) = \int_{-\infty}^{\infty} \frac{\mathcal{F}_{p,k_m}(\Omega)}{i\Omega + Dk^2_{p,m}} e^{i\Omega t} \frac{d\Omega}{2\pi}. \tag{A15}$$

It is easy to find now

$$\langle \tilde{u}^*_{p,k_m}(t) \tilde{u}_{p,k'_m}(t+\tau) \rangle = 4\pi^2 DR \frac{k_B T^2}{C_p V_m \rho} \times \delta(k_m - k'_m) \delta_{p,p'} \int_{-\infty}^{\infty} \frac{e^{i\Omega\tau}}{\Omega^2 + D^2 k^4_{p,m}} \frac{d\Omega}{2\pi}. \tag{A16}$$

Therefore, according to Eq. (A11),

$$S_{\bar{u}}(\Omega) = \frac{k_B T^2}{C_p V_m \rho} \frac{2D}{R} \sum_{p \le \nu^{2/3}} \int_{-\infty}^{\infty} \frac{dk_m}{2\pi} \times \left[\frac{8\pi^2 \sin(k_m L/2)}{k_m L (k_m^2 L^2 - 4\pi^2)} \right]^2 \frac{1}{\Omega^2 + D^2 k^4_{p,m}}. \tag{A17}$$

To check self-consistence of the obtained result we substitute Eq. (A17) into Eq. (8):

$$\langle (\Delta T)^2_m \rangle = \int_{-\infty}^{\infty} S_{\bar{u}}(\Omega) \frac{d\Omega}{2\pi} = \frac{k_B T^2}{C_p V_m \rho} \frac{1}{R} \sum_{p \le \nu^{2/3}} \int_{-\infty}^{\infty} \frac{dk_m}{2\pi} \left[\frac{8\pi^2 \sin(k_m L/2)}{k_m L (k_m^2 L^2 - 4\pi^2)} \right]^2 \frac{1}{k^2_{p,m}}. \tag{A18}$$

To evaluate the sum in Eq. (A18), we assume that $\nu \gg 1$, $k^2_{p,m} = k^2_m + \beta^2_p / R^2 \approx k^2_m + \pi^2 p^2 / R^2 (4R^2/\pi^2 L^2 \gg 1)$, and find that Eq. (A18) coincides with Eq. (3).

APPENDIX B: SOLUTION OF THE STOCHASTIC HEAT TRANSFER EQUATION FOR A THIN CYLINDRICAL RESONATOR

Eigenfunctions of the heat transfer equation with zero Neumann boundary conditions,

$$\frac{\partial u}{\partial t} - D\Delta u = 0, \tag{B1}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0, \tag{B2}$$

for a cylinder with radius R and height L can be presented in the form

$$u(t) = \sum_{p,l,m} \tilde{u}_{p,l,m}(t) \cos\left(\frac{\pi m z}{L} \right) J_l(k_{l,p} r) e^{\pm i l \phi}, \tag{B3}$$

where $J_l(k_{l,p} r)$ is the Bessel function of the first kind, and $k_{l,p}$ are roots of the equation $\partial J_l(k_{l,p} r)/\partial r|_R = 0$. Equation (B1) describes the temperature modes of a WGM resonator near heat balance. Only modes with $l=0$ and $m=0,2$ should be taken into account if we are interested in a study of thermal properties of optical WGMs with field distribution given by

$$\Psi(\mathbf{r}) = \sin\left(\frac{\pi z}{L} \right) \sqrt{\frac{2}{\pi}} \frac{J_\nu(k_{\nu,q} r)}{J_{\nu+1}(k_{\nu,q} R)} \frac{e^{\pm i\nu\phi}}{R\sqrt{L}}, \tag{B4}$$

where $k_{\nu,q}$ are roots of the equation $J_\nu(k_{\nu,q} R) = 0$, and $\int |\Psi(\mathbf{r})|^2 d\mathbf{r} = 1$. All other temperature modes have zero contribution to the average temperature of the WGM channel [see Eq. (6)].

Hence, we are looking for a solution of Eq. (5) having the form

$$u(t) = \sum_{p,m} \Phi_{p,m}(\mathbf{r}) \tilde{u}_{p,m}(t), \tag{B5}$$

$$\int \Phi_{p,m} \Phi_{p',m'} d\mathbf{r} = \delta_{m,m'} \delta_{p,p'}. \tag{B6}$$

The space-dependent parts of the function can be found from

$$\Phi_{p,0}(\mathbf{r}) = \frac{J_0(k_p r)}{\sqrt{\pi} J_0(k_p R)} \frac{1}{R\sqrt{L}}, \tag{B7}$$

$$\Phi_{p,2}(\mathbf{r}) = \cos\left(\frac{2\pi z}{L} \right) \sqrt{\frac{2}{\pi}} \frac{J_0(k_p r)}{J_0(k_p R)} \frac{1}{R\sqrt{L}}, \tag{B8}$$

where $k_p = \beta_p / R$, β_p is the pth root of the equation $J_1(\beta_p) = 0$. We used the fact that $J'_0(r) = -J_1(r)$. It is also important to note that $\beta_p \simeq \pi p$ if $p \gg 1$.

Substituting Eq. (B5) into Eq. (5), multiplying the equation by $\Phi_{p',m'}(\mathbf{r})$, and averaging over the volume, we obtain

$$\frac{\partial \tilde{u}_{p,m}(t)}{\partial t} + Dk^2_{p,m} \tilde{u}_{p,m}(t) = \mathcal{F}_{p,m}(t), \tag{B9}$$

where $k^2_{p,m} = k^2_p + (\pi m/L)^2$ and

$$\langle \mathcal{F}^*_{p,m}(t) \mathcal{F}_{p',m'}(t') \rangle = 16\pi \frac{k_B T^2 DL}{\rho C V_m} \delta(t - t') \delta_{m,m'} \delta_{p,p'}. \tag{B10}$$

To find $\bar{u}(t)$ [see Eq. (6)], we note that

$$\int \Phi_{p,0}(\mathbf{r}) |\Psi(\mathbf{r})|^2 d\mathbf{r} \simeq \frac{1}{\sqrt{\pi R^2 L}}, \quad \nu^{2/3} \ge p, \tag{B11}$$

$$\int \Phi_{p,2}(\mathbf{r})|\Psi(\mathbf{r})|^2 \mathrm{d}\mathbf{r} \simeq -\frac{1}{\sqrt{\pi R^2 L}}, \quad \nu^{2/3} \geq p, \quad \text{(B12)}$$

where we have assumed that the temperature distribution function can be taken out of the integral if the temperature does not change significantly at a range comparable with the spatial distribution of the WGM field. The spectral power density is

$$S_{\bar{u}}(\Omega) = \frac{1}{\pi R^2 L} \sum_{p \leq \nu^{2/3}} \sum_{m=0,2} \int_{-\infty}^{\infty} \langle \tilde{u}^*_{p,m}(t) \tilde{u}_{p,m}(t+\tau) \rangle e^{-i\Omega\tau} \mathrm{d}\tau. \quad \text{(B13)}$$

To find the spectral density, we solve Eq. (B9) using the Fourier transform. We decompose time-dependent variables as follows:

$$\tilde{u}_{p,m}(t) = \int_{-\infty}^{\infty} \tilde{u}_{p,m}(\Omega) e^{i\Omega t} \frac{\mathrm{d}\Omega}{2\pi}, \quad \text{(B14)}$$

$$\mathcal{F}_{p,m}(t) = \int_{-\infty}^{\infty} \mathcal{F}_{p,m}(\Omega) e^{i\Omega t} \frac{\mathrm{d}\Omega}{2\pi}, \quad \text{(B15)}$$

where

$$\langle \mathcal{F}^*_{p,m}(\Omega) \mathcal{F}_{p',m'}(\Omega') \rangle = 32\pi^2 \frac{k_B T^2 D L}{\rho C V_m} \delta(\Omega - \Omega') \delta_{m,m'} \delta_{p,p'}. \quad \text{(B16)}$$

The solution of Eq. (B9) is

$$\tilde{u}_{p,m}(t) = \int_{-\infty}^{\infty} \frac{\mathcal{F}_{p,m}(\Omega)}{i\Omega + D k^2_{p,m}} e^{i\Omega t} \frac{\mathrm{d}\Omega}{2\pi}. \quad \text{(B17)}$$

It is easy to find now

$$\langle \tilde{u}^*_{p,m}(t) \tilde{u}_{p,m}(t+\tau) \rangle = 16\pi \frac{k_B T^2 D L}{\rho C V_m} \delta_{m,m'} \delta_{p,p'} \int_{-\infty}^{\infty} \frac{e^{i\Omega\tau}}{\Omega^2 + D^2 k^4_{p,m}} \frac{\mathrm{d}\Omega}{2\pi}. \quad \text{(B18)}$$

Therefore, according to Eq. (B13),

$$S_{\bar{u}}(\Omega) = \frac{k_B T^2}{\rho C V_m} \frac{16 D}{R^2} \sum_{p \leq \nu^{2/3}} \sum_{m=0,2} \frac{1}{\Omega^2 + D^2 k^4_{p,m}}. \quad \text{(B19)}$$

To check self-consistence of the obtained result, we substitute Eq. (B19) into Eq. (8):

$$\langle (\Delta T)^2_m \rangle = \int_{-\infty}^{\infty} S_{\bar{u}}(\Omega) \frac{\mathrm{d}\Omega}{2\pi} = \frac{k_B T^2}{\rho C V_m} \frac{8}{R^2} \sum_{p \leq \nu^{2/3}} \sum_{m=0,2} \frac{1}{k^2_{p,m}}. \quad \text{(B20)}$$

To evaluate the sums in Eq. (B20), we have assumed that $\nu \gg 1$ and find

$$\sum_{p \leq \nu^{2/3}} \frac{1}{k^2_{p,0}} = R^2 \sum_{p \leq \nu^{2/3}} \frac{1}{\beta_p^2} \approx R^2 \sum_{p=1}^{\infty} \frac{1}{\beta_p^2} = \frac{R^2}{8}, \quad \text{(B21)}$$

$$\sum_{p \leq \nu^{2/3}} \frac{1}{k^2_{p,2}} \approx \frac{R^2}{\pi^2} \sum_{p=1}^{\infty} \frac{1}{p^2 + 4R^2/\pi^2 L^2} \approx \frac{RL}{4}. \quad \text{(B22)}$$

To evaluate the sum (B22), we also have assumed that $\beta_p \approx \pi p$ for $p \gg 1$, which is a good approximation if $4R^2/\pi^2 L^2 \gg 1$. We see, finally, that Eq. (B20) coincides with Eq. (3) for $R \gg L$.

APPENDIX C: SIMPLIFICATION OF THE EXPRESSION FOR THE SPECTRAL POWER DENSITY

We start from a realistic model and consider a calcium fluoride resonator of radius $R = 0.3$ cm and thickness $L = 0.01$ cm. The resonator is driven with $\lambda = 1.55\ \mu$m light, so $\nu \simeq 2\pi R n/\lambda \approx 2.7 \times 10^4$ and $R/\nu^{2/3} \simeq 1.1 \times 10^{-3}$. The thermal diffusivity for CaF_2 is equal to $D = 3.6 \times 10^{-2}\ \mathrm{cm^2/s}$, hence characteristic frequencies for the process are $D/R^2 = 0.4\ \mathrm{s^{-1}}$, $D\nu^{4/3}/R^2 = 3.2 \times 10^5\ \mathrm{s^{-1}}$, and $D/L^2 = 360\ \mathrm{s^{-1}}$. The evaluated spectral density of the WGM frequency deviation is shown in Fig. 2. To find the factor $k_B T^2/\rho C V_m \simeq 7 \times 10^{-14}\ \mathrm{K^2}$, have assumed that $V_m = 2\pi R L \times R/\nu^{2/3} \simeq 6 \times 10^{-6}\ \mathrm{cm^3}$.

Let us find an analytical approximation Eq. (B19). We evaluate first two asymptotics:

$$S_{\bar{u}}(\Omega)\Big|_{\Omega \ll D/R^2} = \frac{k_B T^2}{\rho C V_m} \frac{R^2}{12 D}, \quad \text{(C1)}$$

$$S_{\bar{u}}(\Omega)\Big|_{\Omega \gg 4D/L^2, D\nu^{4/3}/R^2} = \frac{k_B T^2}{\rho C V_m} \frac{32 D \nu^{2/3}}{R^2 \Omega^2}. \quad \text{(C2)}$$

Using the asymptotics as well as the normalization condition Eq. (3), we obtain Eq. (10). The comparison of the approximation as well as the exact solution is shown in Fig. 8.

APPENDIX D: RADIAL OSCILLATIONS OF A SPHERE

Let us find the amplitude of the radial oscillations of a sphere as well as the uncertainty of the amplitude originating from the thermal motion. For the sake of simplic-

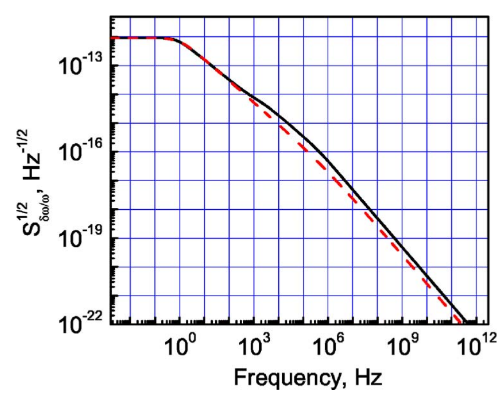

Fig. 8. (Color online) Spectral density of the frequency noise $S_{\delta\omega/\omega}(\Omega) = \alpha_n^2 S_{\bar{u}}(\Omega)$. The solid (black) curve is derived from the exact solution in Eq. (B19). The dashed (red) curve is the analytical approximation given by Eq. (10).

ity, we consider the radial oscillations only. It means that we approximate a solid-state spherical resonator with a liquid one. Adiabatic longitudinal acoustic modes in a liquid sphere are described by the Helmholtz equation,

$$\Delta \mathbf{U} + \Omega^2 \beta_S \rho \mathbf{U} = 0, \tag{D1}$$

where $\mathbf{U}$ is the displacement, β_S is the adiabatic compressibility of the medium, ρ is the density of the medium, and Ω is the frequency. It worth noting that $v_s = (\beta_S \rho)^{-1/2}$ is the speed of sound in the medium.

The solution of the equation under free boundary conditions is

$$\mathbf{U} = \frac{\nabla u_{m,n,p}(\mathbf{r})}{\Omega^2 \beta_S \rho}, \tag{D2}$$

where

$$u_{m,n,p}(\mathbf{r}) = u_0 \sqrt{\frac{\pi}{2}} \frac{J_{n+1/2}(\Omega_{n,p} r (\beta_S \rho)^{1/2})}{[\Omega_{n,p} r (\beta_S \rho)^{1/2}]^{1/2}} P_n^m(\cos \theta) e^{\pm i m \phi}, \tag{D3}$$

where r, θ, and ϕ are the spherical coordinates, u_0 is a scaling factor depending on m, n, and p; $J_{n+1/2}(z)$ is the Bessel function of the first kind, $P_n^m(\cos \theta)$ is an associated Legendre function, $n=0,1,2,3,\ldots$, $m=0,1,2,3,\ldots$, and $p=1,2,3,\ldots$. Eigenfrequencies $\Omega_{n,p}$ are given by the equation

$$J_{n+1/2}(\Omega_{n,p} R (\beta_S \rho)^{1/2}) = 0. \tag{D4}$$

Equations (D3) and (D4) have an especially simple form for the purely radial modes ($m=0$ and $n=0$):

$$u_{0,0,p}(\mathbf{r}) = u_{p0} \frac{\sin(\Omega_{0,p} r (\beta_S \rho)^{1/2})}{\Omega_{0,p} r (\beta_S \rho)^{1/2}}, \tag{D5}$$

$$\sin(\Omega_{0,p} R (\beta_S \rho)^{1/2}) = 0. \tag{D6}$$

The solution of Eq. (D6) is

$$\Omega_{0,p} = \frac{\pi p}{R (\beta_S \rho)^{1/2}}. \tag{D7}$$

The radial displacement for each mode in this case is

$$U_r(p) = u_{p0} \frac{R^2}{\pi^2 p^2} \frac{\partial}{\partial r} \left[\frac{\sin(\Omega_{0,p} r (\beta_S \rho)^{1/2})}{\Omega_{0,p} r (\beta_S \rho)^{1/2}} \right]. \tag{D8}$$

The tensor of deformation has only three nonzero components:

$$U_{rr}(p) = \frac{\partial U_r(p)}{\partial r}, \quad U_{\theta\theta}(p) = U_{\phi\phi}(p) = \frac{U_r(p)}{r}. \tag{D9}$$

The internal energy of the mode p is

$$\begin{aligned} \mathcal{E}_p &= \frac{1}{2\beta_T} \int (U_{rr}(p) + U_{\theta\theta}(p) + U_{\phi\phi}(p))^2 \mathrm{d}V \\ &= \frac{\pi^2}{\beta_T} \int_0^R (U_{rr}(p) + U_{\theta\theta}(p) + U_{\phi\phi}(p))^2 r^2 \mathrm{d}r \\ &= \frac{u_{p0}^2}{2p^2} \frac{R^3}{\beta_T}. \end{aligned} \tag{D10}$$

Assuming that $\langle \mathcal{E}_p \rangle = k_B T/2$, we obtain

$$\langle U_r(p)^2|_R \rangle = \frac{1}{\pi^4 p^2} \frac{k_B T \beta_T}{R}. \tag{D11}$$

Summing over all the modes, we get

$$\frac{\langle (\Delta R)^2 \rangle}{R^2} = \frac{\sum_p \langle U_r(p)^2|_R \rangle}{R^2} = \frac{2}{\pi} \frac{k_B T \beta_T}{9 V_r}, \tag{D12}$$

which nearly corresponds to Eq. (12).

APPENDIX E: SOLUTION OF EQUATION (21) FOR A LOW-CONTRAST RESONATOR

The solution of Eq. (21) is similar to the solution of Eq. (5). Substituting Eq. (A5) into Eq. (21), multiplying the equation by $\Phi^*_{p',k'_m}(\mathbf{r})$, and averaging over the volume, we obtain

$$\frac{\partial \tilde{u}_{p,k_m}(t)}{\partial t} + D k_{p,m}^2 \tilde{u}_{p,k_m}(t) = \mathcal{F}_{p,k_m}(t), \tag{E1}$$

where $k_{p,m}^2 = k_p^3 + k_m^2$ and

$$\langle \mathcal{F}^*_{p,k_m}(t) \mathcal{F}_{p',k'_m}(t') \rangle = g_1(p,p',m,m') \frac{\hbar \omega_{\nu,q,m}}{\rho^2 C^2} \langle P_{abs} \rangle \delta(t-t'),$$

$$g_1(p,p',m,m') = \int |\Psi(\mathbf{r})|^2 \Phi_{p,k_m}(\mathbf{r}) \Phi^*_{p',k'_m}(\mathbf{r}) \mathrm{d}\mathbf{r}.$$

The solution of Eq. (E1) is

$$\tilde{u}_{p,m}(t) = \int_{-\infty}^{\infty} \frac{\mathcal{F}_{p,k_m}(\Omega)}{i\Omega + D k_{p,m}^2} e^{i\Omega t} \frac{\mathrm{d}\Omega}{2\pi}, \tag{E2}$$

$$\begin{aligned} &\langle \tilde{u}^*_{p,k_m}(t) \tilde{u}_{p',k'_m}(t+\tau) \rangle \\ &\quad = \frac{\hbar \omega_{\nu,q,m}}{\rho^2 C^2} \langle P_{abs} \rangle \int_{-\infty}^{\infty} \frac{g_1(p,p',m,m') e^{i\Omega r}}{(-i\Omega + D k_{p,m}^2)(i\Omega + D k_{p',m'}^2)} \frac{\mathrm{d}\Omega}{2\pi}. \end{aligned} \tag{E3}$$

The spectral power density is

$$S_{\bar{u}}(\Omega) = \frac{\hbar\omega_{\nu,q,m}}{\rho^2 C^2}\langle P_{abs}\rangle \sum_{p,p'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \times \frac{\mathrm{d}k_m}{2\pi}\frac{\mathrm{d}k_{m'}}{2\pi}\frac{g_1(p,p',m,m')g_2^*(p,m)g_2(p',m')}{(i\Omega + Dk_{p,m}^2)(-i\Omega + Dk_{p',m'}^2)},$$

$$g_2(p,m) = \int |\Psi(\mathbf{r})|^2 \Phi_{p,k_m}(\mathbf{r})\mathrm{d}\mathbf{r}. \tag{E4}$$

This expression can be further simplified:

$$S_{\bar{u}}(\Omega) \approx \frac{\hbar\omega_{\nu,q,m}}{\rho^2 C^2}\frac{\langle P_{abs}\rangle L^2}{V_r^2}\left| \sum_{p \le \nu^{2/3}} \int_{-\pi/4L}^{\pi/4L} \frac{\mathrm{d}k_m}{i\Omega + Dk_{p,m}^2}\right|^2. \tag{E5}$$

Let us first find the uncertainty of the mode temperature due to the process:

$$\langle(\Delta T)^2\rangle = \int_{-\infty}^{\infty} S_{\bar{u}}(\Omega)\frac{\mathrm{d}\Omega}{2\pi} = \frac{\hbar\omega_{\nu,q,m}}{\rho^2 C^2}\frac{L^2\langle P_{abs}\rangle}{DV_r^2}\sum_{p,p' \le \nu^{2/3}}\int_{-\pi/4L}^{\pi/4L}\int_{-\pi/4L}^{\pi/4L}\frac{\mathrm{d}k_m \mathrm{d}k_m'}{k_{p,m}^2 + k_{p',m'}^2} \simeq \frac{\hbar\omega_{\nu,q,m}}{\rho^2 C^2}\frac{\pi^3}{32}\frac{R^2\langle P_{abs}\rangle}{DV_r^2}. \tag{E6}$$

It is also possible to find

$$S_{\bar{u}}(0) \simeq \frac{\hbar\omega_{\nu,q,m}}{\rho^2 C^2}\frac{\langle P_{abs}\rangle}{V_r^2}\frac{L^{3/2}R^{5/2}}{4D^2}. \tag{E7}$$

We estimate $S_{\delta\omega/\omega}^{1/2}(0) = \alpha_n S_{\bar{u}}^{1/2}(0) \simeq 10^{-15}\,\mathrm{Hz}^{-1/2}$ for a fluorite resonator with $R = 0.3$ cm and $L = 0.01$ cm interrogated with 1.55 μm light of 1 mW power.

To derive the simplified expression for the spectral density, we assume that there is a single characteristic heat transfer time in the system. Keeping in mind Eqs. (E6) and (E7), we obtain Eq. (23).

APPENDIX F: PONDEROMOTIVE FLUCTUATIONS

To find the fluctuations of the resonator radius resulting from the radiation pressure of the light traveling in the resonator, we present the power circulating inside the resonator as $P = \hbar\omega_{\nu,q,m} a^\dagger a/\tau_0$, where $\tau_0 = 2\pi Rn/c$. It worth noting also that the ratio between the internal power (P) circulating in the resonator and the external pump power (P_p) is

$$\frac{P}{P_p} = \frac{2\gamma_R}{\gamma_R + \gamma_{R_0}}\frac{1}{\tau_0(\gamma_R + \gamma_{R_0})}, \tag{F1}$$

where γ_{R_0} is the intrinsic loss rate of the resonator. We assume that the resonator is overcoupled, i.e., $\gamma_R \gg \gamma_{R_0}$.

The annihilation operator a obeys the equation

$$\dot{a} + (\gamma_R + i\omega_{\nu,q,m})a = \gamma_R a_0 e^{-i\omega_{\nu,q,m}t} + \sqrt{2\gamma_R}f(t), \tag{F2}$$

where fluctuational force $f(t)$ has only one nonzero moment $\langle f(t)f^\dagger(t')\rangle = \delta(t-t')$; a_0 is given by the expectation value of the power $\langle P\rangle = \hbar\omega_{\nu,q,m}|a_0|^2/\tau_0$, and γ_R stands for the coupling width of the resonance (we assume that the resonator is overcoupled). The steady-state solution of Eq. (F2) is

$$a = a_0 e^{-i\omega_{\nu,q,m}t} + \int_{-\infty}^{\infty}\frac{\sqrt{2}\gamma_R f(\omega)}{\gamma_R + i(\omega_{\nu,q,m} - \omega)}e^{-i\omega t}\frac{\mathrm{d}\omega}{2\pi}, \tag{F3}$$

where $\langle f(\omega), f^\dagger(\omega')\rangle = 2\pi\delta(\omega - \omega')$. Hence, we derive

$$\delta P(t) = \sqrt{\frac{\hbar\omega_{\nu,q,m}\langle P\rangle}{\tau_0}}\int_{-\infty}^{\infty}\frac{\sqrt{2\gamma_R}}{\gamma_R + i\omega}(f(\omega + \omega_{\nu,q,m}) + f^\dagger(\omega_{\nu,q,m} - \omega))e^{-i\omega t}\frac{\mathrm{d}\omega}{2\pi}. \tag{F4}$$

Presenting the deviation of the radius as

$$\Delta R(t) = \int_{-\infty}^{\infty}\Delta R(\Omega)e^{-i\Omega t}\frac{\mathrm{d}\Omega}{2\pi}, \tag{F5}$$

we get

$$\Delta R(\Omega) = \frac{2\pi n}{m^* c}\sqrt{\frac{\hbar\omega_{\nu,q,m}\langle P\rangle}{\tau_0}} \times \frac{\sqrt{2\gamma_R}}{\gamma_R + i\Omega}\frac{f(\Omega + \omega_{\nu,q,m}) + f^\dagger(\omega_{\nu,q,m} - \Omega)}{\Omega_0^2 - \Omega^2 - i\Gamma_0\Omega}. \tag{F6}$$

It is easy to find now

$$\langle\Delta R^2(t)\rangle = \left(\frac{2\pi n}{m^* c}\right)^2\frac{\hbar\omega_{\nu,q,m}\langle P\rangle}{\tau_0} \times \int_{-\infty}^{\infty}\frac{2\gamma_R}{\gamma_R^2 + \Omega^2}\frac{\mathrm{d}\Omega/2\pi}{(\Omega_0^2 - \Omega^2)^2 + \Gamma_0^2\Omega^2}. \tag{F7}$$

Therefore, the spectral density of the fluctuations and the quadratic deviation of the radius can be represented as those in Eqs. (26) and (27).

ACKNOWLEDGMENTS

The research described in this paper was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration, and with support from DARPA. A. Matsko gratefully acknowledges useful discussions with Dmitry Strekalov.

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